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A note on almost Riemann Solitons and gradient almost Riemann Solitons

Krishnendu De[1] · Uday Chand De[2]

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Abstract
The object of the offering article is to investigate an *almost Riemann soliton* and a *gradient almost Riemann soliton* in a non-cosymplectic normal almost contact metric manifold M^3. Before all else, it is proved that if the metric of M^3 is a Riemann soliton with divergence-free potential vector field Z, then the manifold is quasi-Sasakian and is of constant sectional curvature -λ, provided α, $\beta =$ constant. Also, it is shown that if the metric of M^3 is an almost Riemann Soliton and Z is pointwise collinear with ξ and has constant divergence, then Z is a constant multiple of ξ and the almost Riemann Soliton reduces to a Riemann soliton, provided α, β =constant. Additionally, it is established that if M^3 with α, $\beta =$ constant admits a gradient almost Riemann soliton (γ, ξ, λ), then the manifold is either quasi-Sasakian or is of constant sectional curvature $-(\alpha^2 - \beta^2)$. Finally, we develop an example of M^3 admitting a Riemann soliton.

Keywords 3-dimensional normal almost contact metric manifold · Almost Riemann soliton · Gradient almost Riemann soliton

Mathematics Subject Classification 53C15 · 53C25

1 Introduction

Since Einstein manifolds play an important role in Mathematics and material science, the examination of Einstein manifolds and their speculations is an intriguing point in Riemannian and contact geometry. Lately, various generalizations of Einstein manifolds such as Ricci solitons, gradient Einstein solitons, gradient Ricci solitons, gradient m-quasi Einstein solitons

✉ Krishnendu De
krishnendu.de@outlook.in

Uday Chand De
uc_de@yahoo.com

[1] Department of Mathematics,Kabi Sukanta Mahavidyalaya, The University of Burdwan, Bhadreswar, P.O.-Angus, Hooghly, West Bengal 712221, India

[2] Department of Pure Mathematics, University of Calcutta, Kolkata, West Bengal, India

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(m,ρ)-quasi Einstein solitons on paracontact geometry

Krishnendu De[1 2] and **Uday Chand De** [3]

Abstract. We set the goal to investigate the geometrical properties of (m,ρ)-*quasi Einstein solitons* within the context of paracontact metric manifolds (especially in para-Sasakian, para-cosymplectic and para-Kenmotsu manifolds). Also, we consider a non-trivial example and validate a result of our paper.

AMS Mathematics Subject Classification (2010): 53C15; 53C50; 53C80

Key words and phrases: 3-dimensional para-Sasakian; para-Kenmotsu; para-cosymplectic manifolds; Gradient ρ-Einstein solitons; (m,ρ)-quasi Einstein metric.

1. Introduction

The investigation of paracontact geometry is performing a crucial role in the development of modern differential geometry. It has many connections with the other areas of mathematics and mathematical physics. Due to its broad applications, it became popular among eminent researchers. In 1985, the topic of paracontact metric structures was introduced in [13]. Since then, the properties of paracontact metric manifolds have been studied by many investigators. The significance of paracontact geometry has been indicated minutely in the previous years by various articles ([15], [16], [14]). We may mention ([1], [6], [8], [7], [9], [13]) and the references contained in those for more information about paracontact metric geometry.

The interesting notion, called as *generalized quasi-Einstein metric* is introduced by Catino [3], for studying harmonic Weyl tensor and defined as follows (see [3]):

Let a C^∞ manifold $\mathcal{N}^n$, $n>2$, admits a Riemannian metric g, then the metric g satisfying

$$S+\nabla^2\gamma=\alpha d\gamma\otimes d\gamma+\beta g$$

is called a *generalized quasi-Einstein metric* for some C^∞ functions α, γ and β, where S, ∇^2, d and $\otimes$ denote the Ricci tensor, Hessian operator, exterior derivative of g and tensor product, respectively. In the current article, we consider an (m,ρ)-*quasi Einstein metric* in the 3-dimensional normal almost paracontact metric (briefly, *apm*) manifolds, which is a particular case of *generalized quasi-Einstein metric*, and study its geometrical properties.

[1]Assistant Professor of Mathematics, Kabi Sukanta Mahavidyalaya, The University of Burdwan, Bhadreswar, P.O.-Angus, Hooghly, Pin.712221, West Bengal, India,
e-mail: krishnendu.de67@gmail.com, ORCID iD: orcid.org/0000-0001-6520-4520
[2]Corresponding author
[3]Department of Pure Mathematics, University of Calcutta, 35, B. C. Road, Kol- 700019, West Bengal, India, e-mail: uc_de@yahoo.com, ORCID iD: orcid.org/0000-0002-8990-4609

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INVESTIGATIONS ON A RIEMANNIAN MANIFOLD WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION AND GRADIENT SOLITONS

KRISHNENDU DE[1], UDAY CHAND DE[2], AND AYDIN GEZER[3]

Abstract. This article carries out the investigation of a three-dimensional Riemannian manifold N^3 endowed with a semi-symmetric type non-metric connection. Firstly, we construct a non-trivial example to prove the existence of a semi-symmetric type non-metric connection on N^3. It is established that a N^3 with the semi-symmetric type non-metric connection, whose metric is a gradient Ricci soliton, is a manifold of constant sectional curvature with respect to the semi-symmetric type non-metric connection. Moreover, we prove that if the Riemannian metric of N^3 with the semi-symmetric type non-metric connection is a gradient Yamabe soliton, then either N^3 is a manifold of constant scalar curvature or the gradient Yamabe soliton is trivial with respect to the semi-symmetric type non-metric connection. We also characterize the manifold N^3 with a semi-symmetric type non-metric connection whose metrics are Einstein solitons and m-quasi Einstein solitons of gradient type, respectively.

1. Introduction

In this paper, on a Riemannian manifold N^3, we carry out an investigation of gradient solitons with a semi-symmetric type non-metric connection (briefly, $SSNMC$). Many years ago, on a differentiable manifold, Friedman and Schouten [11] presented the concept of semi-symmetric linear connection. After that in 1932, on a Riemannian manifold, Hayden [15] introduced the notion of metric connection with torsion. In 1970, a systematic investigation of semi-symmetric metric connection which plays a

Key words and phrases. Riemannian manifolds, gradient Ricci solitons, gradient Yamabe solitons, gradient Einstein solitons, m-quasi Einstein solitons.

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m-quasi Einstein Metric and Paracontact Geometry

Krishnendu De, Uday Chand De* and Fatemah Mofarreh

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ABSTRACT

The object of the upcoming article is to characterize paracontact metric manifolds admitting m-quasi Einstein metric. First we establish that if the metric g in a K-paracontact manifold is the m-quasi Einstein metric, then the manifold is of constant scalar curvature. Furthermore, we classify (k, μ)-paracontact metric manifolds whose metric is m-quasi Einstein metric. Finally, we construct a non-trivial example of such a manifold.

Keywords: *m-quasi Einstein metric, K-paracontact metric manifolds, (k, μ)-paracontact metric manifolds.*
AMS Subject Classification (2020): *Primary: 53C15 ; Secondary: 53C25; 53D10.*

1. Introduction

Kaneyuki and Williams [14] introduced a new subclass of contact metric manifolds called paracontact metric manifolds, in 1985. Several authors researched and generalized the manifold after that. Many geometers considered the manifold to be an interesting topic (see, [1], [2], [9], [10], [11],[15]and references therein). Many relationships exist between paracontact geometry and various disciplines of mathematics, mathematical physics, and material sciences. It became popular among notable geometers due to its wide range of uses.

The study of Einstein metric in Riemannian and contact geometry play a significant role in recent geometrical research. Several generalizations of Einstein manifolds have recently been researched, including Ricci solitons, gradient Ricci solitons, generalized quasi-Einstein solitons, and so on. Catino [8] introduced the fascinating concept of generalized quasi-Einstein metric for investigating harmonic Weyl tensor, which is defined as follows (see [8]):

If a C^∞ manifold M^n, $n > 2$, admits a Riemannian metric g satisfying

$$S + \nabla^2\delta = \alpha d\delta \otimes d\delta + \gamma g,$$

for some smooth functions α, δ and γ, is called a generalized quasi-Einstein metric. Here, S denotes the Ricci tensor, d indicates the exterior derivative of g, ∇^2 being the Hessian operator and $\otimes$ is the tensor product, respectively. If $\alpha = \frac{1}{m}$ and $\gamma \in \mathbb{R}$, where m is an integer, then the above metric reduces to a m-quasi Einstein metric. Analogous to the definition in [13], a Riemannian metric g is called a m-quasi Einstein metric if there exists a smooth function $\delta : M^n \to \mathbb{R}$ such that

$$S + \nabla^2\delta - \frac{1}{m} d\delta \otimes d\delta = \gamma g, \tag{1.1}$$

where γ is a constant.

The m-Bakry-Emery Ricci tensor, which is proportional to the metric g and $\gamma =$ constant [17], is expressed as $S + \nabla^2\delta - \frac{1}{m} d\delta \otimes d\delta$. Here, g, the Riemannian metric with constant potential function δ is trivial and therefore, the manifold becomes an Einstein manifold. Moreover, the previous equation produces the gradient Ricci soliton for $m = \infty$. For $m = 1$, m-quasi Einstein metrics restore static metrics. These metrics have been thoroughly examined in general relativity.

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*** Corresponding author***

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Riemann solitons on para-Sasakian geometry

De K.[1], De U.C.[2]

The goal of the present article is to investigate almost Riemann soliton and gradient almost Riemann soliton on 3-dimensional para-Sasakian manifolds. At first, it is proved that if (g, Z, λ) is an almost Riemann soliton on a para-Sasakian manifold M^3, then it reduces to a Riemann soliton and M^3 is of constant sectional curvature -1, provided the soliton vector Z has constant divergence. Besides these, we prove that if Z is pointwise collinear with the characteristic vector field ξ, then Z is a constant multiple of ξ and the manifold is of constant sectional curvature -1. Moreover, the almost Riemann soliton is expanding. Furthermore, it is established that if a para-Sasakian manifold M^3 admits gradient almost Riemann soliton, then M^3 is locally isometric to the hyperbolic space $H^3(-1)$. Finally, we construct an example to justify some results of our paper.

Key words and phrases: para-Sasakian manifold, almost Riemann soliton, gradient almost Riemann soliton.

[1] Kabi Sukanta Mahavidyalaya, University of Burdwan, Bhadreswar, P.O.-Angus, Hooghly, Pin 712221, West Bengal, India
[2] University of Calcutta, 35, Ballygunge Circular Road, Kol-700019, West Bengal, India
E-mail: krishnendu.de@outlook.in (De K.), uc_de@yahoo.com (De U.C.)

Introduction

Since Einstein manifolds perform a significant role in mathematics and physics, the investigation of Einstein manifolds and their generalizations is a fascinating topic in Riemannian and semi-Riemannian geometry. In the last few years, several generalizations of Einstein manifolds such as Ricci soliton, gradient Ricci soliton, gradient Einstein soliton, etc. have been studied. The idea of Ricci flow was introduced by R.S. Hamilton [8] and expressed by $\frac{\partial}{\partial_t} g(t) = -2S(t)$, where S indicates the Ricci tensor.

As a natural generalization, the notion of Riemann flow (see [18]) is expressed by $\frac{\partial}{\partial_t} G(t) = -2Rg(t)$, $G = \frac{1}{2} g \otimes g$, where R is the Riemann curvature tensor and $\otimes$ is *Kulkarni-Nomizu* product, defined as follows (see [1, p. 47])

$$\begin{aligned}(P \otimes Q)(X, Y, U, W) = P(X, W)Q(Y, U) + P(Y, U)Q(X, W)\\ - P(X, U)Q(Y, W) - P(Y, W)Q(X, U),\end{aligned}$$

where P and Q are $(0, 2)$-tensor field.

Similar to Ricci soliton, the interesting concept of Riemann soliton was introduced by I.E. Hirică and C. Udriste [11]. Analogous to I.E. Hirică and C. Udriste [11], a semi-Riemannian metric g on a semi-Riemannian manifold M is said to be a *Riemann soliton* if there exist a C^∞ vector field Z and a real scalar λ such that

$$2R + \lambda g \otimes g + g \otimes \pounds_Z g = 0. \tag{1}$$

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A note on gradient solitons on two classes of almost Kenmotsu manifolds

Krishnendu De*

Department of Mathematics
Kabi Sukanta Mahavidyalaya, The University of Burdwan
Bhadreswar, P.O.-Angus, Hooghly 712221, West Bengal, India
krishnendu.de@outlook.in
krishnendu.de67@gmail.com

Uday Chand De

Department of Pure Mathematics, University of Calcutta
35, Ballygunge Circular Road, Kolkata 700019
West Bengal, India
uc_de@yahoo.com

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The purpose of the paper is to characterize **gradient** (m, ρ)**-quasi Einstein solitons** within the framework of two classes of almost Kenmotsu Manifolds. Finally, we consider an example to justify a result of our paper.

Keywords: $(k, \mu)'$-almost Kenmotsu manifolds; Kenmotsu manifolds; (m, ρ)-quasi Einstein solitons.

Mathematics Subject Classification 2020: 53D50, 53C25, 53C80

1. Introduction

The techniques of contact geometry carried out a significant role in contemporary mathematics and consequently it has become well known among the eminent researchers. Contact geometry has manifested from the mathematical formalism of classical mechanics. This topic has more than one attachment with the alternative regions of differential geometry and outstanding applications in applied fields such as phase space of dynamical systems, mechanics, optics and thermodynamics. In this paper, we investigate gradient solitons with nullity distribution which play a functional role in modern mathematics.

*Corresponding author.

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Perfect fluid spacetimes obeying certain restrictions on the energy-momentum tensor

Krishnendu De[a], Uday Chand De[b]

[a]*Department of Mathematics, Kabi Sukanta Mahavidyalaya, The University of Burdwan Bhadreswar, P.O.-Angus, Hooghly, Pin 712221, West Bengal, India*
[b]*Department of Pure Mathematics, University of Calcutta, 35 Ballygunge Circular Road, Kolkata -700019, West Bengal, India*

Abstract. This paper is concerned with the study of a perfect fluid spacetime endowed with the various forms of the energy-momentum tensor T. We establish that a perfect fluid spacetime endowed with covariant constant energy-momentum tensor represents a dark matter era or the matter content is a perfect fluid spacetime with vanishing vorticity; whereas a perfect fluid spacetime endowed with Codazzi type of T represents a dark matter era or the expansion scalar vanishes, provided α_1 remains invariant under the velocity vector field ρ. Also, we show that a perfect fluid spacetime with pseudo-symmetric energy-momentum tensor represents a dark matter era or a phantom era, provided the velocity vector field annihilates the curvature tensor. Moreover, we characterize T-recurrent and weakly-T symmetric perfect fluid spacetimes with Killing velocity vector field and acquired that the perfect fluid spacetimes represent a radiation era in the first case and a stiff matter for the last one.

1. Introduction

This paper deals with 4-dimensional spacetimes (that is, a connected time-directed Lorentz manifolds) (M, g) whose Ricci tensor is of the form

$$S = \alpha_1.g + \alpha_2.A \otimes A, \tag{1}$$

where $\alpha_i \in C^\infty(M)$ and A is the 1-form metrically associated with a fixed unit (say future-directed) timelike vector field. Evidently, if $\alpha_2 \equiv 0$, then the contracted Bianchi identity and the equation (1) together imply that (M, g) is an Einstein manifold in that case, that is, α_1 is constant. In this sense, (1) generalizes the Einstein condition, which justifies spacetimes satisfying it, is sometimes called quasi-Einstein in the literature.

The energy–momentum tensor T performs a significant role as a matter content of the spacetime in general relativity (briefly, GR) and matter is assumed to be fluid having density, pressure and dynamical and kinematic quantities like velocity, acceleration, vorticity, shear and expansion. In GR theory, the fluid

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ORCID ID: https://orcid.org/0000-0001-6520-4520 (Krishnendu De), https://orcid.org/0000-0002-8990-4609 (Uday Chand De)
Email addresses: krishnendu.de@outlook.in (Krishnendu De), uc_de@yahoo.com (Uday Chand De)

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Almost co-Kähler manifolds and (m,ρ)-quasi-Einstein solitons

Krishnendu De [a], Mohammad Nazrul Islam Khan [b,*], Uday Chand De [c]

[a] Department of Mathematics, Kabi Sukanta Mahavidyalaya, The University of Burdwan, Bhadreswar, P.O.-Angus, Hooghly, Pin 712221, West Bengal, India
[b] Department of Computer Engineering, College of Computer, Qassim University, Buraydah 51452, Saudi Arabia
[c] Department of Pure Mathematics, University of Calcutta, West Bengal, India

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ABSTRACT

The present paper aims to investigate (m,ρ)-quasi-Einstein metrics on almost co-Kähler manifolds $\mathcal{M}$. It is proven that if a (κ,μ)-almost co-Kähler manifold with $\kappa<0$ is (m,ρ)-quasi-Einstein manifold, then $\mathcal{M}$ represents a $N(\kappa)$-almost co-Kähler manifold and the manifold is locally isomorphic to a solvable non-nilpotent Lie group. Next, we study the three dimensional case and get the above mentioned result along with the manifold $\mathcal{M}^3$ becoming an η-Einstein manifold. We also show that there does not exist (m,ρ)-quasi-Einstein structure on a compact (κ,μ)-almost co-Kähler manifold of dimension greater than three with $\kappa<0$. Further, we prove that an almost co-Kähler manifold satisfying η-Einstein condition with constant coefficients reduces to a K-almost co-Kähler manifold, provided $ma_1 \neq (2n-1)b_1$ and $m\neq 1$. We also characterize perfect fluid spacetime whose Lorentzian metric is equipped with (m,ρ)-quasi Einstein solitons and acquired that the perfect fluid spacetime has vanishing vorticity, or it represents dark energy era under certain restriction on the potential function. Finally, we construct an example of an almost co-Kähler manifold with (m,ρ)-quasi-Einstein solitons.

1. Introduction

In [1], Boothby and Wang investigated an odd-dimensional differentiable manifold in 1958 with the help of almost contact and contact structure and investigated its features from a topological perspective. Making use of tensor calculus, Sasaki [2] described the characteristics of a differentiable manifold which is odd-dimensional, with contact structures in 1960. Such manifolds were referred to as contact manifolds. After that, other researchers have discovered various types of contact manifolds and investigated their characteristics. The Kenmotsu, the Sasakian, and the co-Kähler manifolds are Tanno's divisions of the almost contact metric manifolds [3], whose automorphism groups have the highest dimensions. It should be noted that the cosymplectic manifolds that Blair [4] introduced and Goldberg and Yano [5] examined are nothing but the co-Kähler manifolds. The co-Kähler manifolds can be thought of from some topological perspectives as the analogue of Kähler manifolds in odd-dimension, that is why the new terminology has been introduced. The characteristics of almost co-Kähler manifolds, which are an extension of co-Kähler manifolds, were examined by several researchers. Perrone [6] provided a thorough classification of the three-dimensional homogeneous almost co-Kähler manifolds as well as a local characterization of these manifolds under the assumption of local symmetry. We advise readers to read [7–11] and the references therein for more information regarding almost co-Kähler manifolds.

In both mathematics and physics, Einstein manifolds are crucial. In Riemannian and Semi-Riemannian geometry, it is interesting to investigate Einstein manifolds and their generalizations. Numerous generalizations of Einstein manifolds have been developed recently, including quasi-Einstein manifolds [12], generalized quasi-Einstein manifolds [13], m-quasi-Einstein manifolds [10], (m,ρ)-quasi-Einstein manifolds [14], and many more.

A Riemannian metric g of an almost co-Kähler manifold is named a Ricci soliton [15] if there is a $\lambda_1 \in \mathbb{R}=$ constant, $\mathbb{R}$ is the set of all real numbers and a smooth vector field X such that

$$Ric + \frac{1}{2}\pounds_X g = \lambda_1 g, \tag{1.1}$$

in which Ric indicates the Ricci tensor of the metric tensor g and $\pounds$ denotes the Lie-derivative. If λ_1 is a smooth function, then the above soliton is called an almost Ricci soliton [16].

If we choose a smooth function $\omega : \mathcal{M}\to\mathbb{R}$ with $X=D\omega$, D being the gradient operator of g, then it is named a gradient Ricci soliton. Hence, Eq. (1.1) reduces to

$$Ric + Hess\,\omega = \lambda_1 g, \tag{1.2}$$

where $Hess$ is the Hessian operator.

A Ricci soliton is nothing but a natural generalization of Einstein metric [17]. In this paper we study (m,ρ)-quasi-Einstein solitons which are the generalization of Einstein solitons and gradient Ricci solitons.

* Corresponding author.
E-mail addresses: krishnendu.de@outlook.in (K. De), m.nazrul@qu.edu.sa (M.N.I. Khan), ucdpm@caluniv.ac.in (U.C. De).

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Characterizations of a Spacetime of Quasi-Constant Sectional Curvature and $\mathcal{F}(\mathcal{R})$-Gravity

Uday Chand De, Krishnendu De, Fusun Ozen Zengin, and Sezgin Altay Demirbag*

The main aim of this article is to investigate a spacetime of quasi-constant sectional curvature. At first, the existence of such a spacetime is established by several examples. We have shown that a spacetime of quasi-constant sectional curvature agrees with the present state of the universe and it represents a Robertson Walker spacetime. Moreover, if the spacetime is Ricci semi-symmetric or Ricci symmetric, then either the spacetime represents a spacetime of constant sectional curvature, or the spacetime represents phantom era. Also, we prove that a Ricci symmetric spacetime of quasi-constant sectional curvature represents a static spacetime and the spacetime under consideration is of Petrov type I, D or O. Finally, we concentrate on a quasi-constant sectional curvature spacetime solution in $\mathcal{F}(\mathcal{R})$-gravity. As a result, various energy conditions are studied and analyzed our obtained outcomes in terms of a $\mathcal{F}(\mathcal{R})$-gravity model.

1. Introduction

To study a conformally flat hypersurfaces of a Euclidean space the authors[10] acquire the ensuing expression of the curvature tensor

$$R_{hijk} = \gamma(g_{hk}g_{ij} - g_{hj}g_{ik}) + \mu\{g_{hk}A_iA_j + g_{ij}A_hA_k - g_{hj}A_iA_k - g_{ik}A_hA_j\}, \tag{1.1}$$

in which γ, μ are scalars and A_i is a unit vector, called the generator. An n-dimensional conformally flat space obeying (1.1) is named a space of quasi-constant sectional curvature and denoted by $(QC)_n$.

U. C. De
Department of Pure Mathematics
University of Calcutta
West Bengal, India

K. De
Department of Mathematics
Kabi Sukanta Mahavidyalaya
The University of Burdwan
P.O.-Angus, Hooghly, Bhadreswar, West Bengal Pin 712221, India
E-mail: krishnendu.de@outlook.in

F. O. Zengin, S. A. Demirbag
Department of Mathematics
Istanbul Technical University
Istanbul 34469, Turkey

The ORCID identification number(s) for the author(s) of this article can be found under https://doi.org/10.1002/prop.202200201

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However, if the equation (1.1) of the curvature tensor holds, then it can be easily verified that the space is conformally flat. So in the definition conformally flatness is not required. A space of quasi-constant sectional curvature has been studied by several authors, such as ([4, 10, 22, 37]) and many others.

Spacetime is a time-oriented 4-dimensional Lorentzian manifold $\mathcal{M}$ which is a specific class of semi-Riemannian manifold endowed with a Lorentzian metric g of signature $(-,+,+,+)$. A Lorentzian manifold is named a spacetime of quasi-constant sectional curvature if the curvature tensor R_{hijk} fulfills the condition (1.1). Here, we assume the generator A_i is a unit time-like vector field, that is, $A_iA^i = -1$, $A^i = g^{ij}A_j$.

From the foregoing definition we can easily acquire

$$R_{ij} = \{(n-1)\gamma + \mu\}g_{ij} + (n-2)\mu A_iA_j; \quad A_iA^i = -1, \tag{1.2}$$

where $R_{hk} = R^i_{hki}$ is the Ricci tensor.

Multiplying the above equation with g^{ij} yields

$$\mathcal{R} = n(n-1)\gamma - 2(n-1)\mu, \tag{1.3}$$

where $\mathcal{R} = g^{hk}R_{hk}$ is the Ricci scalar. Again multiplying (1.2) by A^iA^j, we get

$$R_{ij}A^iA^j = -(n-1)\gamma + (n-3)\mu. \tag{1.4}$$

Let us denote the Ricci curvature $R_{ij}A^iA^j$ by ν.

Hence from equations (1.3) and (1.4), we obtain

$$\gamma = \frac{(n-3)\mathcal{R} - 2\nu}{(n-1)\{n(n-3)+2\}}; \quad \mu = \frac{\mathcal{R} + n\nu}{n(n-3)+2}. \tag{1.5}$$

Therefore, the associated scalars γ and μ are related with the Ricci scalar $\mathcal{R}$ and Ricci curvature ν.

In a spacetime $\mathcal{M}^n$, for a smooth function $\psi > 0$ (also called scale factor, or warping function), if $\mathcal{M} = -I\times_\psi$ M, where I is the open interval of $\mathbb{R}$, M^{n-1} denotes the Riemannian manifold, then $\mathcal{M}$ is named a generalized Robertson Walker (briefly, GRW) spacetime.[1] If $\mathcal{M}^3$ is of constant sectional curvature, then the spacetime represents a Robertson Walker (briefly, RW) spacetime.

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Ricci-Yamabe Solitons in $f(R)$-gravity

Krishnendu De and Uday Chand De*

(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

The main objective of this paper is to describe the perfect fluid spacetimes fulfilling $f(R)$-gravity, when Ricci-Yamabe, gradient Ricci-Yamabe and η-Ricci-Yamabe solitons are its metrics. We acquire conditions for which the Ricci-Yamabe and the gradient Ricci-Yamabe solitons are expanding, steady or shrinking. Furthermore, we investigate η-Ricci-Yamabe solitons and deduce a Poisson equation and with the help of this equation, we acquire some significant results.

Keywords: *Perfect fluids; $f(R)$-gravity; Ricci-Yamabe solitons; η-Ricci-Yamabe solitons.*
AMS Subject Classification (2020): *Primary: 83D05; Secondary: 83C05; 53C50.*

1. Introduction

Einstein's field equations are insufficient to describe the universe late-time inflation without presuming that there are some unobserved components that could explain the origin of dark energy and dark matter. It is the main inspiration behind the extension to find field equations of gravity in higher order. It is noteworthy that the concept of $f(R)$-gravity emerges as an inherent extension of Einstein's theory of gravity. Here, the Hilbert-Einstein action term is changed by the function $f(R)$, in which R stands for the Ricci scalar. Buchdahl proposed the aforementioned hypothesis [6] and Starobinsky [27] has accomplished viability with the study of cosmic inflation. In earlier works, a number of useful functional forms of $f(R)$ were presented in ([7], [8], [9], [14], [15], [18]).

Spacetime is a time-oriented 4-dimensional Lorentzian manifold $\mathcal{M}$ which is a specific class of semi-Riemannian manifold endowed with a Lorentzian metric g of signature $(-,+,+,+)$.

The Einstein's field equation is described in the general theory of relativity by

$$Ric - \frac{R}{2}g = \kappa^2 T, \tag{1.1}$$

in which $\kappa^2 = 8\pi G$, G denotes Newton's gravitational constant and $'Ric'$ stands for the Ricci tensor. The preceding equation entails that the energy momentum tensor T has vanishing divergence.

In a perfect fluid-spacetime the energy momentum tensor T is described by,

$$T = pg + (\nu + p)\mathcal{D} \otimes \mathcal{D}, \tag{1.2}$$

in which $\mathcal{D}$ is a 1-form, p stands for the isotropic pressure, ν denotes the energy density of the perfect fluid-spacetime. Also, $g(\rho,\rho) = -1$, since here ρ is a unit timelike vector field, named the velocity vector, defined by $g(E,\rho) = \mathcal{D}(E)$ for any E.

The far more fascinating mathematical techniques for illustrating the geometric structures in differential geometry throughout the most recent years are those based on the theory of geometric flows. Due to the fact that they emerge as convincing singularity models, the analysis of flow singularities is significantly affected by a certain portion of solutions where the metric changes through diffeomorphisms. These are commonly known as soliton solutions.

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**** Corresponding author***

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Perfect fluid spacetimes and k-almost Yamabe solitons

Krishnendu DE[1], **Uday Chand DE**[2], **Aydin GEZER**[3,*]
[1]Department of Mathematics, Kabi Sukanta Mahavidyalaya,
The University of Burdwan, Bhadreswar, P.O.-Angus, Hooghly, Pin, West Bengal, India
[2]Department of Pure Mathematics, University of Calcutta, West Bengal, India
[3]Department of Mathematics, Atatürk University, Erzurum, Turkey

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Abstract: In this article, we presumed that a perfect fluid is the source of the gravitational field while analyzing the solutions to the Einstein field equations. With this new and creative approach, here we study k-almost Yamabe solitons and gradient k-almost Yamabe solitons. First, two examples are constructed to ensure the existence of gradient k-almost Yamabe solitons. Then we show that if a perfect fluid spacetime admits a k-almost Yamabe soliton, then its potential vector field is Killing if and only if the divergence of the potential vector field vanishes. Besides, we prove that if a perfect fluid spacetime permits a k-almost Yamabe soliton (g, k, ρ, λ), then the integral curves of the vector field ρ are geodesics, the spacetime becomes stationary and the isotopic pressure and energy density remain invariant under the velocity vector field ρ. Also, we establish that if the potential vector field is pointwise collinear with the velocity vector field and $\rho(a) = 0$ where a is a scalar, then either the perfect fluid spacetime represents a phantom era, or the potential function Φ is invariant under the velocity vector field ρ. Finally, we prove that if a perfect fluid spacetime permits a gradient k-almost Yamabe soliton $(g, k, D\Phi, \lambda)$ and R, λ, k are invariant under ρ, then the vorticity of the fluid vanishes.

Key words: Perfect fluids, k-almost Yamabe solitons, Robertson-Walker spacetimes

1. Introduction

In this article, we will deal with the spacetimes that obey the Einstein field equations (in short, EFEs) when a perfect fluid (in short, PF) serves as the source of the gravitational field. If a Lorentzian manifold's nonvanishing Ricci tensor S fulfills the conditions

$$S = \alpha g + \beta C \otimes C, \tag{1.1}$$

it is referred to as a PF spacetime, in which α, β (not simultaneously zero) are scalars, and for any X_1, $g(X_1, \rho) = C(X_1)$, ρ stands for the velocity vector field.

Let (M^n, g) be a Lorentzian manifold whose metric g is of signature $(-, +, +, \ldots, +)$, that is, g is of index 1. In [1], L. Alias et al. proposed the notion of generalized Robertson-Walker (in short, GRW) spacetimes. A GRW spacetime is a Lorentzian manifold M^n $(n \geq 3)$ that may be expressed as $M = -I \times f^2 M^*$, in which the open interval I contained in $\mathbb{R}$, $M^{*(n-1)}$ denotes the Riemannian manifold and $f > 0$ is a smooth function,

*Correspondence: aydingzr@gmail.com
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INVESTIGATION ON GRADIENT SOLITONS IN PERFECT FLUID SPACETIMES

Krishnendu De*

Department of Mathematics,
Kabi Sukanta Mahavidyalaya, University of Burdwan,
Bhadreswar, P.O.-Angus, Hooghly, Pin 712221, West Bengal, India
ORCID iD: https://orcid.org/0000-0001-6520-4520
(e-mail: krishnendu.de@outlook.in)

and

Uday Chand De

Department of Pure Mathematics, University of Calcutta,
35 Ballygunge Circular Road, Kolkata 700019, West Bengal, India
ORCID iD: https://orcid.org/0000-0002-8990-4609
(e-mail: uc_de@yahoo.com)

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This article concerns the study of perfect fluid spacetimes equipped with different types of gradient solitons. It is shown that if a perfect fluid spacetime with Killing velocity vector admits a τ-Einstein soliton of gradient type, then the spacetime represents phantom regime, or ψ remains invariant under the velocity vector field ρ. Besides, we establish that in a perfect fluid spacetime with constant scalar curvature, if the Lorentzian metric is the gradient τ-Einstein soliton, then either the τ-Einstein gradient potential function is pointwise collinear with ρ, or the spacetime represents stiff matter fluid. Furthermore, we prove that under certain conditions, a perfect fluid spacetime turns into a generalized Robertson–Walker spacetime, as well as a static spacetime and such a spacetime is of Petrov type I, D or O. We also characterize perfect fluid spacetimes whose Lorentzian metric is equipped with gradient m-quasi Einstein solitons and that the perfect fluid spacetime has vanishing expansion scalar, or it represents dark energy era under certain restriction on the potential function.

Keywords: gradient τ-Einstein soliton, gradient m-quasi Einstein soliton, perfect fluid spacetime.

1. Introduction

Let $(\mathcal{N}^n, g)$ be a Lorentzian manifold whose metric g is of signature $(+,+,\dots,+,-)$, that is, g is of index 1. In [1], Alias, Romero and Sanchez proposed the concept of generalized Robertson–Walker (briefly, GRW) spacetimes in 1995. A GRW spacetime is a Lorentzian manifold $\mathcal{N}^n$ $(n \geq 3)$ that may be

*Corresponding author.

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Characterizations of perfect fluid spacetimes obeying $f(\mathcal{R})$-gravity equipped with different gradient solitons

Krishnendu De*

Department of Mathematics, Kabi Sukanta Mahavidyalaya
The University of Burdwan, Bhadreswar, P. O.-Angus
Hooghly, 712221 West Bengal, India
krishnendu.de@outlook.in

Young Jin Suh

Department of Mathematics and RIRCM
Kyungpook National University, Daegu 41566, South Korea
yjsuh@knu.ac.kr

Uday Chand De

Department of Mathematics
University of Calcutta, West Bengal, India
uc_de@yahoo.com

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The prime object of this paper is to study the perfect fluid spacetimes obeying $f(\mathcal{R})$-gravity, when η-Ricci solitons, gradient η-Ricci solitons, gradient Einstein solitons and gradient m-quasi Einstein solitons are its metrics. At first, the existence of the η-Ricci solitons is proved by a non-trivial example. We establish conditions for which the η-Ricci solitons are expanding, steady or shrinking. Besides, in the perfect fluid spacetimes obeying $f(\mathcal{R})$-gravity, when the potential vector field of η-Ricci soliton is of gradient type, we acquire a Poisson equation. Moreover, we investigate gradient η-Ricci solitons, gradient Einstein solitons and gradient m-quasi Einstein solitons in $f(\mathcal{R})$-gravity, respectively. As a result, we establish some significant theorems about dark matter era.

Keywords: $f(\mathcal{R})$-gravity; perfect fluids; η-Ricci solitons; Einstein solitons; m-quasi Einstein solitons.

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1. Introduction

A Lorentzian manifold is a specific type of semi-Riemannian manifold that has the Lorentzian metric g. A time-oriented four-dimensional Lorentzian manifold $\mathcal{M}$ with signature $(-,+,+,+)$ is called spacetime.

*Corresponding author.